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A GENERALIZATION OF SHEAR DEFORMATION THEORIES FOR AXISYMMETRIC MULTILAYERED SHELLS

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Abstract—A generalization of geometrically linear shear deformation theories for small elastic strains is presented for multilayered axisymmetric shells of general shape without any assumption other than neglecting the transverse normal strain. The shear is taken into account by using a function $f(\zeta)$ which is introduced in the assumed kinematics. The boundary value problem is derived from the principle of virtual power. With the "shear" function $f(\zeta)$ in the kinematics, all equations are directly applicable to: Kirchhoff–Love, first-order shear deformation, third-order shear deformation theories and, obviously, the proposed generalized shear deformation theory by using a certain sine shear function. No shear-correction factors are needed with the proposed generalization of shear deformation theories. A numerical evaluation of the new theory is presented and compared with the above classic theories for a simply-supported thick laminated cylindrical shell under an internal pressure.

INTRODUCTION

In practical applications, shell structures have been most commonly designed as axisymmetric in shape, and a few, such as piping; pressure vessels, ranging from small, high-pressure storage bottles to large chemical storage tanks and ballistic-missile rocket-motor casings, are fully axisymmetric (shape and loading). Using theories or finite elements of axisymmetric shape is thus more efficient than using theories or finite elements of general shells to simulate the behaviour of such structures.

An increasing number of structural designs are extensively utilizing fiber composite laminates. As in the case of laminated plates, the major higher-order effects in composite-material shells are thickness–shear flexibility and thickness–normal stresses. Also, due to low transverse shear moduli relative to in-plane Young's moduli, transverse shear deformation effects are even more pronounced in composite laminates, except perhaps for hygrothermal analysis. Likewise three general approaches are used to analyse these effects: improved shell theory; microstructural continuum shell theory; three-dimensional nonhomogeneous elasticity theory for long hollow cylinders. As examples within the wide field of shell theories, laminated shell theories, shell finite elements, are the contributions of Bert (1974), Naghdi (1971), Bert and Francis (1974), Bhimaraddi (1985), Di Sciuva (1987) and Yang *et al.* (1990). It would seem that the works published on axisymmetric shells are few.

Because of difficulties involved in deriving two-dimensional theories of shells from three-dimensional equations of elasticity, assumptions of one kind or another must customarily be introduced into the derivation. The aim of this paper is to propose a generalization of shear deformation theories for multilayered moderately thick axisymmetric shells. The shear is taken into account by introducing into the kinematics the shear function $f(\zeta) = (h/\pi)\sin(\pi\zeta/h)$, where ζ is the coordinate following the thickness h of the shell. The corresponding boundary value problem is solved by using the principle of virtual power in linear elasticity. All equations are presented with a general shear function $f(\zeta)$ which allows the users to deduce "in extenso" displacements, strains, equilibrium equations, boundary conditions and the constitutive law; for Kirchhoff–Love $f(\zeta) = 0$, first-order shear deformation $f(\zeta) = \zeta$, third-order shear deformation $f(\zeta) = \zeta(1-4\zeta^2/3h^2)$ and generalized shear deformation $f(\zeta) = (h/\pi)\sin(\pi\zeta/h)$ theories, for axisymmetric shells of general shape. A numerical evaluation of theories is presented for a layered cylindrical shell under internal pressure in statics. The computation of stress distributions seems to indicate the superiority of the proposed generalized shear deformation theory. The theory which is geometrically linear, is developed for small elastic strains.

The interest of such a work is in building a simple and efficient moderately thick axisymmetric shell theory for the sizing of pressure vessels and piping, to analyse the collapse modes of axisymmetric shells and to develop finite element approximation for general cases such as arbitrary meridional shapes and nonlinear analysis.

GEOMETRICAL PRELIMINARIES

The fundamental problem of the theory of thin elastic shells is the formulation of a two-dimensional system of differential equations and boundary conditions, for a rational approximate determination of stresses and deformations in three-dimensional elastic bodies shaped as a thin elastic layer surrounding a surface in space, the middle surface of the shell.

Consider the space surrounding an arbitrary surface A, hereafter designated the shell middle surface, which is defined by two curvilinear orthogonal coordinates (ξ_1, ξ_2) coinciding with its lines of principal curvature. Let $\vec{e}_1$ and $\vec{e}_2$ be the unit vectors in the directions ξ_1 and ξ_2, respectively:

$$\vec{e}_1 = \alpha_1^{-1}\frac{\partial \vec{r}}{\partial \xi_1} \quad \text{and} \quad \vec{e}_2 = \alpha_2^{-1}\frac{\partial \vec{r}}{\partial \xi_2}$$

where $\vec{r} = \vec{r}(\xi_1, \xi_2)$ is the position vector of a point on the reference surface. Coefficients:

$$\alpha_1^2 = \frac{\partial \vec{r}}{\partial \xi_1}\cdot\frac{\partial \vec{r}}{\partial \xi_1}; \quad \alpha_2^2 = \frac{\partial \vec{r}}{\partial \xi_2}\cdot\frac{\partial \vec{r}}{\partial \xi_2}$$

are those of the first fundamental form of the shell reference surfaces (surface metric coefficients). The unit vector perpendicular to A is denoted by $\vec{n}$, which is chosen so that $(\vec{e}_1, \vec{e}_2, \vec{n})$ form a right-handed orthogonal system, $\vec{n} = \vec{e}_1 \wedge \vec{e}_2$. The radii of curvature in the directions of ξ_1 and ξ_2 are denoted by R_1 and R_2, respectively, and are taken to be positive when the centers of curvature lie in the negative direction of $\vec{n}$.

Let ζ be a rectilinear coordinate measured along the normal $\vec{n}$ to A. Then, from the surface geometry, we define the square of a line element through the middle surface A:

$$(\mathrm{d}S)^2 = L_1^2(\mathrm{d}\xi_1)^2 + L_2^2(\mathrm{d}\xi_2)^2 + (\mathrm{d}\zeta)^2,$$

and the volume element

$$\mathrm{d}v = L_1 L_2\, \mathrm{d}\xi_1\, \mathrm{d}\xi_2\, \mathrm{d}\zeta$$

where

$$L_1 = \alpha_1\left(1 + \frac{\zeta}{R_1}\right); \quad L_2 = \alpha_2\left(1 + \frac{\zeta}{R_2}\right)$$

are the Lamé parameters or the coefficients of the second fundamental form of the shell middle surface. For a point on the middle surface, because $\zeta = 0$, we have

$$L_1 = \alpha_1; \quad L_2 = \alpha_2$$

and the corresponding square of a line element is (the first fundamental form of the shell middle surface)

$$(ds)^2 = \alpha_1^2(d\xi_1)^2+\alpha_2^2(d\xi_2)^2.$$

GENERAL KINEMATICS FOR AXISYMMETRIC SHELLS OF ARBITRARY SHAPE

Let U_1, U_2, U_ζ be the displacement components at an arbitrary point (ξ_1, ξ_2, ζ) and in the direction of $(\vec{e}_1, \vec{e}_2, \vec{n})$. The displacement components are functions of ξ_1, ξ_2, ζ. For an arbitrary shell and a geometrically linear theory, the strain–displacement relations (kinematic relations) have been given by Reissner (1966) in curvilinear coordinates:

$$\varepsilon_{11} = \frac{1}{1+\dfrac{\zeta}{R_1}}\left(\frac{1}{\alpha_1}\frac{\partial U_1}{\partial \xi_1}+\frac{\partial \alpha_1}{\partial \xi_2}\frac{U_2}{\alpha_1\alpha_2}+\frac{U_\zeta}{R_1}\right)$$

$$2\varepsilon_{12} = \frac{1}{1+\dfrac{\zeta}{R_1}}\left(\frac{1}{\alpha_1}\frac{\partial U_2}{\partial \xi_1}-\frac{\partial \alpha_1}{\partial \xi_2}\frac{U_1}{\alpha_1\alpha_2}\right)+\frac{1}{1+\dfrac{\zeta}{R_2}}\left(\frac{1}{\alpha_2}\frac{\partial U_1}{\partial \xi_2}-\frac{\partial \alpha_2}{\partial \xi_1}\frac{U_2}{\alpha_1\alpha_2}\right)$$

$$2\varepsilon_{1\zeta} = \frac{1}{1+\dfrac{\zeta}{R_1}}\left(\frac{1}{\alpha_1}\frac{\partial U_\zeta}{\partial \xi_1}-\frac{U_1}{R_1}\right)+\frac{\partial U_1}{\partial \zeta}$$

$$\varepsilon_{\zeta\zeta} = \frac{\partial U_\zeta}{\partial \zeta} \tag{1}$$

with corresponding expressions for ε_{22} and $\varepsilon_{2\zeta}$.

For an axisymmetric shell where $(\vec{n}, \vec{e}_1)$ is the plane of symmetry, we have

$$U_2 = 0 \quad \text{and} \quad \partial/\partial\xi_2 = 0. \tag{2}$$

Equation (1) then becomes

$$\varepsilon_{11} = \frac{1}{1+\dfrac{\zeta}{R_1}}\left(\frac{1}{\alpha_1}\frac{\partial U_1}{\partial \xi_1}+\frac{U_\zeta}{R_1}\right)$$

$$\varepsilon_{22} = \frac{1}{1+\dfrac{\zeta}{R_2}}\left(\frac{d\alpha_2}{d\xi_1}\frac{U_1}{\alpha_1\alpha_2}+\frac{U_\zeta}{R_2}\right)$$

$$2\varepsilon_{1\zeta} = \frac{1}{1+\dfrac{\zeta}{R_1}}\left(\frac{1}{\alpha_1}\frac{\partial U_\zeta}{\partial \xi_1}-\frac{U_1}{R_1}\right)+\frac{\partial U_1}{\partial \zeta}$$

$$\varepsilon_{\zeta\zeta} = \frac{\partial U_\zeta}{\partial \zeta}. \tag{3}$$

It is possible that it will be required that all formulations of the two-dimensional theory (still to be established) be deductions from a three-dimensional formulation of a relevant boundary value problem, for instance by an asymptotic expansion of the displacement field. But it is also possible, and very much more practical, to establish a major part of two-dimensional shell theory without reference to any three-dimensional formulation. In this last case, an approximate theory can be constructed on two-dimensional ad hoc assumptions which should lead to the same or nearly the same results as the deductive steps from three to two-dimensions. Then, we assume an approximate displacement field of the following form for an axisymmetric shell:

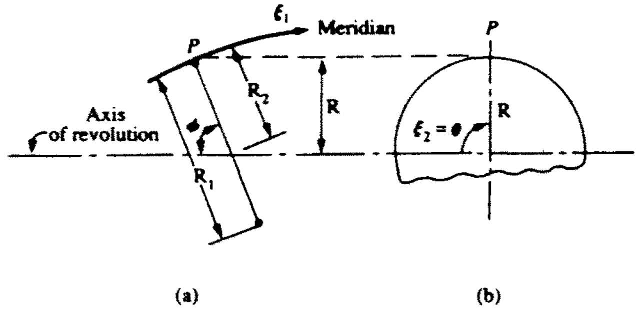

Fig. 1. Geometric variables for an arbitrary doubly curved shell of revolution: (a) meridional plane; (b) cross-section at point P.

$$
\begin{aligned}
U_1 &= \frac{L_1}{\alpha_1}u(\xi_1,t)+\frac{f(\zeta)-\zeta}{\alpha_1}\frac{\partial w}{\partial \xi_1}+f(\zeta)\omega(\xi_1,t)\\
U_2 &= 0\\
U_\zeta &= w(\xi_1,t)
\end{aligned}
\tag{4}
$$

where u and w are displacements of a point P of the meridian, Fig. 1 (respectively following meridional and transverse directions), ω is the relative rotation of the cross-sections to the meridian around the ξ_2-axis and t the time and $\partial w/\partial \xi_1$ is the Kirchhoff–Love rotation of the cross-sections to the meridian also around the ξ_2-axis.

Then, writing U_1 into eqn (4) by taking

$$\gamma = \omega+\frac{1}{\alpha_1}\frac{\partial w}{\partial \xi_1}, \tag{5}$$

we obtain

$$
\begin{aligned}
U_1 &= \frac{L_1}{\alpha_1}u(\xi_1,t)-\frac{\zeta}{\alpha_1}\frac{\partial w}{\partial \xi_1}+f(\zeta)\gamma(\xi_1,t)\\
U_2 &= 0\\
U_\zeta &= w(\xi_1,t).
\end{aligned}
\tag{6}
$$

This last equation shows that the function $f(\zeta)$ is associated with the transverse shear, since $\gamma(\xi_1,t)$ is the shear rotation.

The motives for assuming general kinematics under the form showed by eqn (4) are:

—we want only three independent generalized displacements,
—classical finite element approximations must be possible,
—the transverse shear is taken into account, without shear correction factors if $f(\zeta)$ is a higher-order function of the thickness coordinate ζ,
—as usual in structural mechanics, we suppose a zero transverse normal strain ($\varepsilon_{\zeta\zeta}=0$), which is a good assumption, except perhaps for hygrothermal effects, or for a sandwich shell in which the transverse rigidity of the core is small compared to that of skins,
—the necessity of recovering the classical thin shell and first-order shear deformation theories.

There is no exact three-dimensional elasticity solution for shells, and *two-dimensional appropriate assumptions must be made in order to deduce an efficient theory*, i.e. utilization simplicity, accuracy, viable finite element approximations, shear deformation without correction factors, no higher-order derivatives in the kinematics (these involve some complications to clarify the edge conditions), kinematics independent of the material behaviour in order to extend the theory in plasticity.

Some further comments will be made in the next section when we explain the function $f(\zeta)$ and clarify eqn (4). Before this, we will compute strains to expose the boundary value problem and the constitutive law with an arbitrary function $f(\zeta)$.

So, from eqn (3), the strains associated with the kinematics defined by eqn (4) are for any function $f(\zeta)$:

$$\varepsilon_{11} = \frac{1}{1+\dfrac{\zeta}{R_1}}\left[\frac{-\mathrm{d}R_1}{\mathrm{d}\xi_1}\frac{\zeta}{\alpha_1 R_1^2}u + \frac{L_1}{\alpha_1^2}\frac{\partial u}{\partial \xi_1} + \frac{f(\zeta)-\zeta}{\alpha_1^2}\frac{\partial^2 w}{\partial \xi_1^2} - \frac{f(\zeta)-\zeta}{\alpha_1^3}\frac{\mathrm{d}\alpha_1}{\mathrm{d}\xi_1}\frac{\partial w}{\partial \xi_1} \right.$$
$$\left. + \frac{f(\zeta)}{\alpha_1}\frac{\partial \omega}{\partial \xi_1} + \frac{w}{R_1}\right]$$

$$\varepsilon_{22} = \frac{1}{1+\dfrac{\zeta}{R_2}}\left[\frac{L_1}{\alpha_1^2\alpha_2}\frac{\mathrm{d}\alpha_2}{\mathrm{d}\xi_1}u + \frac{f(\zeta)-\zeta}{\alpha_1^2\alpha_2}\frac{\mathrm{d}\alpha_2}{\mathrm{d}\xi_1}\frac{\partial w}{\partial \xi_1} + \frac{f(\zeta)}{\alpha_2\alpha_1}\frac{\mathrm{d}\alpha_2}{\mathrm{d}\xi_1}\omega + \frac{w}{R_2}\right]$$

$$2\varepsilon_{1\zeta} = \frac{1}{1+\dfrac{\zeta}{R_1}}\left[\frac{1}{\alpha_1}\frac{\partial w}{\partial \xi_1} - \frac{L_1}{\alpha_1}\frac{u}{R_1} - \frac{f(\zeta)-\zeta}{\alpha_1 R_1}\frac{\partial w}{\partial \xi_1} - f(\zeta)\frac{\omega}{R_1}\right]$$
$$+ \frac{u}{R_1} + \left(\frac{\mathrm{d}f}{\mathrm{d}\zeta} - 1\right)\frac{1}{\alpha_1}\frac{\partial w}{\partial \xi_1} + \frac{\mathrm{d}f}{\mathrm{d}\zeta}\omega. \tag{7}$$

The last equation in eqns (7), which represents the transverse shear strain, can still be written after some rearrangements:

$$2\varepsilon_{1\zeta} = \left[\frac{\mathrm{d}f}{\mathrm{d}\zeta} - \frac{f(\zeta)}{R_1\left(1+\dfrac{\zeta}{R_1}\right)}\right]\gamma(\xi_1, t). \tag{8}$$

THE BOUNDARY VALUE PROBLEM FOR AXISYMMETRIC SHELLS OF ARBITRARY SHAPE: THE GENERAL THEORY

A geometrically linear theory for small elastic strains is discussed. The theory is restricted to axisymmetric shells under axisymmetric loading and classical boundary conditions, but is developed for any function $f(\zeta)$.

The shell considered has a uniform thickness which is much smaller than the shell's radii of curvature. The shell may be composed of a single material or several different materials bonded together in layers, each layer having a constant thickness. Each layer may be isotropic or orthotropic. The material properties are assumed to be linearly elastic. A consistent combination of displacements (essential boundary conditions), forces and moments (natural boundary conditions) are specified at the ends of the shell; the upper and lower lines $\zeta = \pm h/2$ being under a normal pressure. So, given the initial geometry of the shell, its material properties, the prescribed end forces and displacements, the displacements and stresses at every point of the shell are required. They are, in fact, obtained by solving a two-dimensional boundary value problem, or a one-dimensional boundary value problem in axisymmetric cases. This boundary value problem may be formulated by a displacement variational method such as: the principle of virtual work, the principle of virtual power. We choose the principle of virtual power [for example see Germain (1986)].

Let Ω be a shell with tractions F prescribed along part of its boundary $\Gamma_\sigma \subset \Gamma_{\mathrm{edge}}$, and displacements prescribed along the other part $\Gamma_u \subset \Gamma_{\mathrm{edge}}$, where the symbol $\subset$ denotes a subset. The upper and lower surfaces of the shell are taken to be under a normal pressure.

To establish the boundary value problem by the principle of virtual power, from eqn (4), we start by defining two spaces $\mathscr{U}$ and $\mathscr{U}^*$ such that

$$\mathscr{U} = \left\{ U_1 = \frac{L_1}{\alpha_1} u + \frac{f(\zeta)-\zeta}{\alpha_1}\frac{\partial w}{\partial \xi_1} + f(\zeta)\omega; \quad U_2 = 0; \quad U_\zeta = w; \right.$$

$(u,\omega) \in H^1(\xi_1)xH^1(\xi_1)$; $w \in H^2(\xi_1)$, u, ω, w and $\frac{\partial w}{\partial \xi_1}$ specified on $\Gamma_u \subset \Gamma_{edge}$

(essential boundary conditions) (9)

$$\mathscr{U}^* = \left\{ U_1^* = \frac{L_1}{\alpha_1} U^* + \frac{f(\zeta)-\zeta}{\alpha_1}\frac{\mathrm{d}W^*}{\mathrm{d}\xi_1} + f(\zeta)\Omega^*; \quad U_2^* = 0; \quad U_\zeta^* = W^*; \right.$$

$(U^*, \Omega^*) \in H^1(\xi_1)xH^1(\xi_1)$; $W^* \in H^2(\xi_1)$, $\mathbf{U}^*$ is zero on $\Gamma_u \subset \Gamma_{edge}$ (10)

where $H^1(\xi_1)$ and $H^2(\xi_1)$ are Sobolev spaces. The space $\mathscr{U}$ is the space of admissible displacements $\mathbf{U}(\xi_1, \zeta, t)$ defined in eqn (4), and the space $\mathscr{U}^*$ is the space of virtual velocities $\mathbf{U}^*(\xi_1, \zeta)$ which must be considered at a fixed time. We note that Γ_{edge} is simply the ends of the axisymmetric shell.

The principle of virtual power states that:

"find $(u, w, \omega) \in \mathscr{U}$ so that for every $(U^*, W^*, \Omega^*) \in \mathscr{U}^*$, we have

$$\int_\Omega \rho \ddot{\mathbf{U}}\mathbf{U}^* \,\mathrm{d}v = -\int_\Omega \sigma_{ij} D_{ij}^* \,\mathrm{d}v + \int_\Omega \hat{\mathbf{f}}\mathbf{U}^* \,\mathrm{d}v + \int_{\Gamma_\sigma} \hat{\mathbf{F}}\mathbf{U}^* \,\mathrm{d}a, \tag{11}$$

with a summation on i and $j = 1, 2, 3$."

In eqn (11), σ is the stress tensor, D^* the virtual strain rate tensor, ρ the mass density, $\ddot{\mathbf{U}} = \partial^2\mathbf{U}/\partial t^2$ the acceleration vector, $\hat{\mathbf{f}}$ the body force vector, $\hat{\mathbf{F}}$ the contact force vector prescribed on Γ_σ which is located at the ends of the shell in the axisymmetric case. The virtual velocity $\mathbf{U}^*$ is defined by eqn (10), while D_{ij}^* is exactly defined as in eqn (7), by substituting ε by D^*, u by U^*, w by W^* and ω by Ω^*, in eqn (7). Equation (11) allows us to obtain both equilibrium equations and boundary conditions. But it is still necessary to develop the integrals in eqn (11).

The integral of the first member in eqn (11) is the acceleration virtual power $\mathscr{P}_a^*$ and is equal by definition to

$$\mathscr{P}_a^* = \int_\Omega \rho \ddot{\mathbf{U}}\mathbf{U}^* \,\mathrm{d}v$$

or, explicitly

$$\mathscr{P}_a^* = \int_0^1 \int_0^{2\pi} \int_{-h/2}^{+h/2} \rho \left[\left(\frac{L_1}{\alpha_1}\ddot{u} + \frac{f(\zeta)-\zeta}{\alpha_1}\frac{\partial \ddot{w}}{\partial \xi_1} + f(\zeta)\ddot{\omega} \right)\left(\frac{L_1}{\alpha_1} U^* + \frac{f(\zeta)-\zeta}{\alpha_1}\frac{\mathrm{d}W^*}{\mathrm{d}\xi_1} + f(\zeta)\Omega^* \right) + \ddot{w}w^* \right] L_1 L_2 \,\mathrm{d}\xi_1 \,\mathrm{d}\xi_2 \,\mathrm{d}\zeta, \tag{12}$$

where $L_\beta = (1+\zeta/R_\beta)\alpha_\beta$, $\beta = 1, 2$ and h is the constant shell thickness.

As all functions in eqn (12) are independent of ξ_2 (the axisymmetric case) and are known functions of the ζ thickness-variable, then eqn (12) becomes

$$\mathscr{P}_a^* = 2\pi \int_0^1 \left[\left(I_u \ddot{u} + \frac{I_{uw}}{\alpha_1} \frac{\partial \ddot{w}}{\partial \xi_1} + I_{u\omega} \ddot{\omega} \right) U^* + \frac{1}{\alpha_1} \left(I_{uw} \ddot{u} + \frac{I_w}{\alpha_1} \frac{\partial \ddot{w}}{\partial \xi_1} + I_{\omega w} \ddot{\omega} \right) \frac{\mathrm{d}W^*}{\mathrm{d}\xi_1} \right.$$
$$\left. + \left(I_{u\omega} \ddot{u} + \frac{I_{\omega w}}{\alpha_1} \frac{\partial \ddot{w}}{\partial \xi_1} + I_{\omega} \ddot{\omega} \right) \Omega^* + I_w \ddot{w} W^* \right] \alpha_1 \alpha_2 \, \mathrm{d}\xi_1 \quad (13)$$

where

$$(I_u, I_{uw}, I_{u\omega}, I_w, I_{\omega w}, I_{\omega}, I_w) = \int_{-h/2}^{+h/2} \rho \left[\left(\frac{L_1}{\alpha_1} \right)^2, \frac{L_1}{\alpha_1} (f(\zeta) - \zeta), \frac{L_1}{\alpha_1} f(\xi), \right.$$
$$\left. (f(\zeta) - \zeta)^2, f(\zeta)(f(\zeta) - \zeta), (f(\zeta))^2, 1 \right] \mathrm{d}\zeta. \quad (14)$$

The first term of the second member of eqn (11) is the internal virtual power $\mathscr{P}_i^*$: $\mathscr{P}_i^* = -\int_\Omega \sigma_{ij} D_{ij}^* \, \mathrm{d}v$. From the definition of the virtual strain rates D_{ij}^* recalled above, the internal virtual power becomes, in axisymmetric cases,

$$\mathscr{P}_i^* = -2\pi \int_0^1 \int_{-h/2}^{+h/2} \left[\frac{\sigma_{11}}{L_1/\alpha_1} \left(\frac{-\zeta}{\alpha_1 R_1^2} \frac{\mathrm{d}R_1}{\mathrm{d}\xi_1} U^* + \frac{L_1}{\alpha_1^2} \frac{\mathrm{d}U^*}{\mathrm{d}\xi_1} + \frac{f(\zeta) - \zeta}{\alpha_1^2} \frac{\mathrm{d}^2 W^*}{\mathrm{d}\xi_1^2} - \frac{f(\zeta) - \zeta}{\alpha_1^3} \frac{\mathrm{d}\alpha_1}{\mathrm{d}\xi_1} \frac{\mathrm{d}W^*}{\mathrm{d}\xi_1} \right. \right.$$
$$\left. + \frac{f(\zeta)}{\alpha_1} \frac{\mathrm{d}\Omega^*}{\mathrm{d}\xi_1} + \frac{W^*}{R_1} \right) + \frac{\sigma_{22}}{L_2/\alpha_2} \left(\frac{L_1}{\alpha_1^2 \alpha_2} \frac{\mathrm{d}\alpha_2}{\mathrm{d}\xi_1} U^* + \frac{f(\zeta) - \zeta}{\alpha_1^2 \alpha_2} \frac{\mathrm{d}\alpha_2}{\mathrm{d}\xi_1} \frac{\mathrm{d}W^*}{\mathrm{d}\xi_1} \right.$$
$$\left. \left. + \frac{f(\zeta)}{\alpha_1 \alpha_2} \frac{\mathrm{d}\alpha_2}{\mathrm{d}\xi_1} \Omega^* + \frac{W^*}{R_2} \right) + \sigma_{1\zeta} \left(\frac{\mathrm{d}f}{\mathrm{d}\zeta} - \frac{f(\zeta)}{R_1 L_1/\alpha_1} \right) \left(\frac{1}{\alpha_1} \frac{\mathrm{d}W^*}{\mathrm{d}\xi_1} + \Omega^* \right) \right] L_1 L_2 \, \mathrm{d}\xi_1 \, \mathrm{d}\zeta, \quad (15)$$

where we recall that

$$\alpha_\beta = \alpha_\beta(\xi_1), \quad L_\beta = \alpha_\beta(1 + \zeta/R_\beta), \quad R_\beta = R_\beta(\xi_1), \quad \beta = 1, 2.$$

From eqn (15), we define the following generalized stresses:

$$N_{11} = \int_{-h/2}^{+h/2} \sigma_{11} \frac{L_2}{\alpha_2} \mathrm{d}\zeta, \quad N_{22} = \int_{-h/2}^{+h/2} \sigma_{22} \frac{L_1}{\alpha_1} \mathrm{d}\zeta$$
$$M_{11} = \int_{-h/2}^{+h/2} \zeta \sigma_{11} \frac{L_2}{\alpha_2} \mathrm{d}\zeta, \quad M_{22} = \int_{-h/2}^{+h/2} \zeta \sigma_{22} \frac{L_1}{\alpha_1} \mathrm{d}\zeta$$
$$\tilde{M}_{11} = \int_{-h/2}^{+h/2} f(\zeta) \sigma_{11} \frac{L_2}{\alpha_2} \mathrm{d}\zeta, \quad \tilde{M}_{22} = \int_{-h/2}^{+h/2} f(\zeta) \sigma_{22} \frac{L_1}{\alpha_1} \mathrm{d}\zeta$$
$$\tilde{Q}_{1\zeta} = \int_{-h/2}^{+h/2} \left(\frac{\mathrm{d}f}{\mathrm{d}\zeta} - \frac{f(\zeta)}{R_1 L_1/\alpha_1} \right) \sigma_{1\zeta} \frac{L_1}{\alpha_1} \frac{L_2}{\alpha_2} \mathrm{d}\zeta \quad (16)$$

which are the classical membrane forces $N_{\beta\beta}$, flexural moments $M_{\beta\beta}$ and other (refined or higher-order) moments $\tilde{M}_{\beta\beta}$ due to the shear function $f(\zeta)$ and the shear force $\tilde{Q}_{1\zeta}$. We note:

—with the Kirchhoff–Love theory, $f(\zeta) = 0$ and $\tilde{M}_{\beta\beta} = 0$, $\tilde{Q}_{1\zeta} = 0$,
—with the first-order shear deformation theory, $f(\zeta) = \zeta$ and $\tilde{M}_{\beta\beta} \equiv M_{\beta\beta}$.

Thus, $\tilde{M}_{\beta\beta}$ are in fact higher-order moments only found with refined theories.

These definitions [eqn (16)] allow us to simplify eqn (15):

$$\mathscr{P}_i^* = -2\pi\int_0^1\left[\left(\frac{-M_{11}}{\alpha_1 R_1^2}\frac{dR_1}{d\xi_1}+\frac{N_{22}}{\alpha_1\alpha_2}\frac{d\alpha_2}{d\xi_1}+\frac{M_{22}}{\alpha_1\alpha_2 R_1}\frac{d\alpha_2}{d\xi_1}\right)U^* - \frac{\tilde{M}_{11}-M_{11}}{\alpha_1^3}\frac{d\alpha_1}{d\xi_1}\frac{dW^*}{d\xi_1}\right.$$
$$+\left(\frac{N_{11}}{\alpha_1}+\frac{M_{11}}{\alpha_1 R_1}\right)\frac{dU^*}{d\xi_1}+\left(\frac{N_{11}}{R_1}+\frac{N_{22}}{R_2}\right)W^*+\left(\frac{\tilde{M}_{22}-M_{22}}{\alpha_1^2\alpha_2}\frac{d\alpha_2}{d\xi_1}+\frac{\tilde{Q}_{1\zeta}}{\alpha_1}\right)\frac{dW^*}{d\xi_1}$$
$$\left.+\frac{\tilde{M}_{11}-M_{11}}{\alpha_1^2}\frac{d^2W^*}{d\xi_1^2}+\left(\frac{\tilde{M}_{22}}{\alpha_1\alpha_2}\frac{d\alpha_2}{d\xi_1}+\tilde{Q}_{1\zeta}\right)\Omega^*+\frac{\tilde{M}_{11}}{\alpha_1}\frac{d\Omega^*}{d\xi_1}\right]\alpha_1\alpha_2\,d\xi_1. \quad (17)$$

Finally, we need to explain the two other virtual powers in eqn (11): respectively the body force virtual power $\mathscr{P}_b^*$ and the contact force virtual power $\mathscr{P}_c^*$. The first is equal by definition to

$$\mathscr{P}_b^* = \int_\Omega \hat{\mathbf{f}}\mathbf{U}^*\,dv,$$

or, explicitly

$$\mathscr{P}_b^* = 2\pi\int_0^1\int_{-h/2}^{+h/2}\left[\hat{f}_1\left(\frac{L_1}{\alpha_1}U^*+\frac{f(\zeta)-\zeta}{\alpha_1}\frac{dW^*}{d\xi_1}+f(\zeta)\Omega^*\right)+\hat{f}_\zeta W^*\right]L_1L_2\,d\xi_1\,d\zeta. \quad (18)$$

Finally, the contact force virtual power is

$$\mathscr{P}_c^* = \int_{\Gamma_\sigma}\hat{\mathbf{F}}\mathbf{U}^*\,da,$$

or, explicitly

$$\mathscr{P}_c^* = 2\pi\int_{-h/2}^{+h/2}\left[\hat{F}_1\left(\frac{L_1}{\alpha_1}U^*+\frac{f(\zeta)-\zeta}{\alpha_1}\frac{dW^*}{d\xi_1}+f(\zeta)\Omega^*\right)+\hat{F}_\zeta W^*\right]L_2\,d\zeta \quad (19)$$

since the shell is axisymmetric, and where $\hat{f}_1, \hat{f}_\zeta$ are components of body forces, $\hat{F}_1, \hat{F}_\zeta$ are components of contact (or end) forces, both in curvilinear coordinates (ξ_1, ξ_2, ζ). We establish:

$$n_1 = \int_{-h/2}^{+h/2}\hat{f}_1\left(\frac{L_1}{\alpha_1}\right)^2\frac{L_2}{\alpha_2}\,d\zeta$$
$$\tilde{m}_1 = \int_{-h/2}^{+h/2}\hat{f}_1 f(\zeta)\frac{L_1}{\alpha_1}\frac{L_2}{\alpha_2}\,d\zeta$$
$$q_\zeta = \int_{-h/2}^{+h/2}\hat{f}_\zeta\frac{L_1}{\alpha_1}\frac{L_2}{\alpha_2}\,d\zeta$$
$$m_1 = \int_{-h/2}^{+h/2}\hat{f}_1\zeta\frac{L_1}{\alpha_1}\frac{L_2}{\alpha_2}\,d\zeta. \quad (20)$$

Then eqn (18) becomes

$$\mathscr{P}_b^* = 2\pi\int_0^1\left(n_1U^*+\frac{\tilde{m}_1-m_1}{\alpha_1}\frac{dW^*}{d\xi_1}+\tilde{m}_1\Omega^*+q_\zeta W^*\right)\alpha_1\alpha_2\,d\xi_1. \quad (21)$$

We define equivalent quantities for contact force virtual power $\mathscr{P}_c^*$ at the ends of the shell:

$$T_1 = \int_{-h/2}^{+h/2} \hat{F}_1 \frac{L_2}{\alpha_2} \mathrm{d}\zeta$$

$$\tilde{C}_1 = \int_{-h/2}^{+h/2} \hat{F}_1 f(\zeta) \frac{L_2}{\alpha_2} \mathrm{d}\zeta$$

$$T_\zeta = \int_{-h/2}^{+h/2} \hat{F}_\zeta \frac{L_2}{\alpha_2} \mathrm{d}\zeta$$

$$C_1 = \int_{-h/2}^{+h/2} \hat{F}_1 \zeta \frac{L_2}{\alpha_2} \mathrm{d}\zeta. \tag{22}$$

So, contact (end) force virtual power can be written as

$$\mathscr{P}_c^* = 2\pi\left[\left(\left(T_1 + \frac{C_1}{R_1}\right)U^* + \frac{\tilde{C}_1 - C_1}{\alpha_1}\frac{\mathrm{d}W^*}{\mathrm{d}\xi_1} + \tilde{C}_1\Omega^* + T_\zeta W^*\right)\alpha_2\right]_0^1. \tag{23}$$

Finally, eqn (11) is equivalent to

$$\mathscr{P}_a^* = \mathscr{P}_i^* + \mathscr{P}_b^* + \mathscr{P}_c^*, \quad \forall (U^*, \Omega^*, W^*) \in \mathscr{U}^* \tag{11 bis}$$

and to verify the principle of virtual power, some integrations by part are necessary to eliminate the derivatives on the virtual velocity field $\mathbf{U}^* \in \mathscr{U}^*$ in eqns (13), (17) and (21). After these classic manipulations, eqn (11), i.e. eqn (11 bis) implies (α_1 and $\alpha_2 \neq 0$):

Equilibrium equations for all $U^*(\xi_1)$, $W^*(\xi_1)$ and $\Omega^*(\xi_1)$

$$A_u = \frac{M_{11}}{\alpha_1 R_1^2}\frac{\mathrm{d}R_1}{\mathrm{d}\xi_1} - \frac{N_{22}}{\alpha_1\alpha_2}\frac{\mathrm{d}\alpha_2}{\mathrm{d}\xi_1} - \frac{M_{22}}{\alpha_1\alpha_2 R_1}\frac{\mathrm{d}\alpha_2}{\mathrm{d}\xi_1} + \frac{1}{\alpha_1^2\alpha_2}\frac{\partial}{\partial\xi_1}\left\{\left(N_{11} + \frac{M_{11}}{R_1}\right)\alpha_1\alpha_2\right\} + n_1$$

$$\begin{aligned}
A_w = &-\frac{N_{11}}{R_1} - \frac{N_{22}}{R_2} + \frac{1}{\alpha_1^2\alpha_2}\frac{\partial}{\partial\xi_1}\left\{\left(\frac{\tilde{M}_{22} - M_{22}}{\alpha_1\alpha_2}\frac{\mathrm{d}\alpha_2}{\mathrm{d}\xi_1} + \tilde{Q}_{1\zeta}\right)\alpha_1\alpha_2\right\} \\
&- \frac{1}{\alpha_1^2\alpha_2}\frac{\partial^2}{\partial\xi_1^2}\{(\tilde{M}_{11} - M_{11})\alpha_2\} - \frac{1}{\alpha_1^2\alpha_2}\frac{\partial}{\partial\xi_1}\left\{(\tilde{M}_{11} - M_{11})\frac{\mathrm{d}\alpha_1}{\mathrm{d}\xi_1}\frac{\alpha_2}{\alpha_1}\right\} + \bar{q}_\zeta \\
&- \frac{1}{\alpha_1^2\alpha_2}\frac{\partial}{\partial\xi_1}\{(\tilde{m}_1 - m_1)\alpha_1\alpha_2\}
\end{aligned}$$

$$A_\omega = -\frac{\tilde{M}_{22}}{\alpha_1\alpha_2}\frac{\mathrm{d}\alpha_2}{\mathrm{d}\xi_1} - \tilde{Q}_{1\zeta} + \frac{1}{\alpha_1^2\alpha_2}\frac{\partial}{\partial\xi_1}\{\tilde{M}_{11}\alpha_1\alpha_2\} + \tilde{m}_1. \tag{24}$$

The normal pressure, p, acting on the lines $\zeta = \pm h/2$ is included by $\bar{q}_\zeta = q_\zeta + p$ in eqn (24).

Natural boundary conditions at the ends $\xi_1 = 0$ and $\xi_1 = 1$ of the shell for all U^*, W^*, $\mathrm{d}W^*/\mathrm{d}\xi_1$, Ω^* so that $\xi_1 = 0$ or $\xi_1 = 1$

$$\begin{aligned}
0 &= T_1 + \frac{C_1}{R_1} - \left(N_{11} + \frac{M_{11}}{R_1}\right)\alpha_1 \\
\bar{A}_w &= T_\zeta + \alpha_1(\tilde{m}_1 - m_1) + \left(\frac{\partial\tilde{M}_{11}}{\partial\xi_1} - \frac{\partial M_{11}}{\partial\xi_1}\right) + \frac{1}{\alpha_2}\frac{\mathrm{d}\alpha_2}{\mathrm{d}\xi_1}(\tilde{M}_{11} - M_{11}) \\
&\quad - \frac{1}{\alpha_2}\frac{\mathrm{d}\alpha_2}{\mathrm{d}\xi_1}(\tilde{M}_{22} - M_{22}) + \frac{1}{\alpha_1}\frac{\mathrm{d}\alpha_1}{\mathrm{d}\xi_1}(\tilde{M}_{11} - M_{11}) - \tilde{Q}_{1\zeta}\alpha_1 \\
0 &= (\tilde{C}_1 - C_1) - (\tilde{M}_{11} - M_{11})\alpha_1 \\
0 &= \tilde{C}_1 - \tilde{M}_{11}\alpha_1.
\end{aligned} \tag{25}$$

In eqns (24) and (25) we have included from eqn (13):

$$A_u = I_u\ddot{u} + \frac{I_{uw}}{\alpha_1}\frac{\partial \ddot{w}}{\partial \xi_1} + I_{u\omega}\ddot{\omega}$$

$$A_w = I_w\ddot{w} - \frac{1}{\alpha_2\alpha_1^2}\frac{\partial}{\partial \xi_1}\left\{\left(I_{uw}\,\ddot{u} + \frac{I_w}{\alpha_1}\frac{\partial \ddot{w}}{\partial \xi_1} + I_{\omega w}\,\ddot{\omega}\right)\alpha_1\alpha_2\right\}$$

$$A_\omega = I_{u\omega}\ddot{u} + \frac{I_{\omega w}}{\alpha_1}\frac{\partial \ddot{w}}{\partial \xi_1} + I_\omega\ddot{\omega}$$

$$\bar{A}_w = \left(I_{uw}\ddot{u} + \frac{I_w}{\alpha_1}\frac{\partial \ddot{w}}{\partial \xi_1} + I_{\omega w}\ddot{\omega}\right)\alpha_1. \tag{26}$$

Equation for $\bar{A}_w$ in eqn (26) is defined only at $\xi_1 = 0$ and $\xi_1 = 1$.

Remarks

(a) We obtain the Kirchhoff–Love theory with $\tilde{M}_{\beta\beta} = \tilde{m}_1 = \tilde{C}_1 = 0$ and $\tilde{Q}_{1\zeta} = 0$ in eqns (24) and (25); $f(\zeta) = 0$.

(b) We obtain the first-order shear deformation theory with $\tilde{M}_{\beta\beta} = M_{\beta\beta}$, $\tilde{m}_1 = m_1$, $\tilde{C}_1 = C_1$ in eqns (24) and (25); $f(\zeta) = \zeta$.

(c) Essential boundary conditions have been defined in the space $\mathcal{U}$, eqn (9).

THE CONSTITUTIVE LAW FOR MULTILAYERED AXISYMMETRIC SHELLS WITHIN THE GENERAL THEORY

Let us consider a shell of constant thickness h consisting of N parallel thin layers of orthotropic linear elastic materials. The thickness of each layer is assumed to be constant. The material properties and the thickness of each layer may be entirely different. As usual in structural mechanics, the normal transverse stress $\sigma_{\zeta\zeta}$ is assumed to be small in comparison with other normal stresses and is neglected. Except for local effect problems, this hypothesis is well justified. Taking into account this assumption and eliminating $\sigma_{\zeta\zeta}$ in the usual anisotropic linear-elastic constitutive law $\sigma_{ij} = C_{ijkl}\varepsilon_{kl}$ (with summation on k and l of 1–3), where C_{ijkl} are the classic elastic coefficients, then, for each layer of an axisymmetric shell (with which we also have $\sigma_{12} = \sigma_{2\zeta} = 0$ and $\varepsilon_{12} = \varepsilon_{2\zeta} = 0$) the constitutive relations become

$$\begin{Bmatrix}\sigma_{11}\\ \sigma_{22}\\ \sigma_{1\zeta}\end{Bmatrix} = \begin{bmatrix}C'_{1111} & C'_{1122} & 0\\ C'_{1122} & C'_{2222} & 0\\ 0 & 0 & C_{1313}\end{bmatrix}\begin{Bmatrix}\varepsilon_{11}\\ \varepsilon_{22}\\ 2\varepsilon_{1\zeta}\end{Bmatrix} \tag{27}$$

where $C'_{\alpha\alpha\beta\beta} = C_{\alpha\alpha\beta\beta} - C_{\alpha\alpha33}C_{\beta\beta33}/C_{3333}$; $\alpha, \beta = 1, 2$, takes into account $\sigma_{\zeta\zeta} = 0$. Shear correction factors are only needed with the first-order shear deformation theory as discussed in the next section.

To use equilibrium equations (24), it is necessary to build a global constitutive law with eqns (16) and (27). Substituting eqn (7) into eqn (27), then the result into eqn (16) gives the global constitutive law:

$$\begin{Bmatrix}\mathbf{N}\\ \mathbf{M}\\ \tilde{\mathbf{M}}\end{Bmatrix} = \begin{bmatrix}\mathbf{A} & \mathbf{B} & \mathbf{A}' & \tilde{\mathbf{B}}\\ \mathbf{B}' & \mathbf{D} & \mathbf{B} & \mathbf{d}\\ \tilde{\mathbf{B}}' & \mathbf{d} & \tilde{\mathbf{B}} & \tilde{\mathbf{D}}\end{bmatrix}\begin{Bmatrix}\varepsilon_u\\ \varepsilon_{uw}\\ \varepsilon_w\\ \varepsilon_{\omega w}\end{Bmatrix} \tag{28}$$

and

$$\tilde{Q}_{1\zeta} = \tilde{A}_{55}\left(\frac{1}{\alpha_1}\frac{\partial w}{\partial \xi_1} + \omega\right). \tag{29}$$

In eqn (28), we have noted:

$$\varepsilon_u^{\mathrm{T}} = \left\{\frac{1}{\alpha_1}\frac{\partial u}{\partial \xi_1}, \frac{\mathrm{d}\alpha_2}{\mathrm{d}\xi_1}\frac{u}{\alpha_1\alpha_2}\right\};$$

$$\varepsilon_{uw}^{\mathrm{T}} = \left\{\frac{-\mathrm{d}R_1}{\mathrm{d}\xi_1}\frac{u}{\alpha_1 R_1^2} - \frac{1}{\alpha_1^2}\frac{\partial^2 w}{\partial \xi_1^2} + \frac{1}{\alpha_1^3}\frac{\mathrm{d}\alpha_1}{\mathrm{d}\xi_1}\frac{\partial w}{\partial \xi_1}, \frac{-1}{\alpha_1^2\alpha_2}\frac{\mathrm{d}\alpha_2}{\mathrm{d}\xi_1}\frac{\partial w}{\partial \xi_1}\right\}$$

$$\varepsilon_w^{\mathrm{T}} = \left\{\frac{w}{R_1}, \frac{w}{R_2}\right\};$$

$$\varepsilon_{\omega w}^{\mathrm{T}} = \left\{\frac{1}{\alpha_1^2}\frac{\partial^2 w}{\partial \xi_1^2} + \frac{1}{\alpha_1}\frac{\partial \omega}{\partial \xi_1} - \frac{1}{\alpha_1^3}\frac{\mathrm{d}\alpha_1}{\mathrm{d}\xi_1}\frac{\partial w}{\partial \xi_1}, \frac{1}{\alpha_1\alpha_2}\frac{\mathrm{d}\alpha_2}{\mathrm{d}\xi_1}\left(\frac{1}{\alpha_1}\frac{\partial w}{\partial \xi_1} + \omega\right)\right\} \tag{30}$$

and

$$\mathbf{P}^{\mathrm{T}} = \{P_{11}, P_{22}\} \quad \text{with} \quad \mathbf{P} = \mathbf{N}, \mathbf{M}, \tilde{\mathbf{M}} \tag{31}$$

$$\tilde{A}_{55} = \int_{-h/2}^{+h/2} C_{1111}\left(\frac{\mathrm{d}f}{\mathrm{d}\zeta} - \frac{f(\zeta)}{R_1 L_1/\alpha_1}\right)^2 \frac{L_1}{\alpha_1}\frac{L_2}{\alpha_2}\,\mathrm{d}\zeta. \tag{32}$$

The matrices $\mathbf{A}$, $\mathbf{B}$, $\mathbf{A}'$, $\mathbf{B}'$, $\tilde{\mathbf{B}}$, $\tilde{\mathbf{B}}'$, $\mathbf{d}$, $\mathbf{D}$, $\tilde{\mathbf{D}}$ in eqn (28) are symmetric and defined by

$$(A_{11}, A_{12}, A_{22}) = \int_{-h/2}^{+h/2}\left(C'_{1111}\frac{L_2}{\alpha_2}, C'_{1122}\frac{L_1}{\alpha_1}, C'_{2222}\frac{L_1^2}{\alpha_1^2}\frac{\alpha_2}{L_2}\right)\mathrm{d}\zeta$$

$$(B_{11}, B_{12}, B_{22}) = \int_{-h/2}^{+h/2}\zeta\left(C'_{1111}\frac{L_2}{\alpha_2}\frac{\alpha_1}{L_1}, C'_{1122}, C'_{2222}\frac{L_1}{\alpha_1}\frac{\alpha_2}{L_2}\right)\mathrm{d}\zeta$$

$$(A'_{11}, A'_{12}, A'_{22}) = \int_{-h/2}^{+h/2}\left(C'_{1111}\frac{L_2}{\alpha_2}\frac{\alpha_1}{L_1}, C'_{1122}, C'_{2222}\frac{L_1}{\alpha_1}\frac{\alpha_2}{L_2}\right)\mathrm{d}\zeta$$

$$(B'_{11}, B'_{12}, B'_{22}) = \int_{-h/2}^{+h/2}\zeta\left(C'_{1111}\frac{L_2}{\alpha_2}, C'_{1122}\frac{L_1}{\alpha_1}, C'_{2222}\frac{L_1^2}{\alpha_1^2}\frac{\alpha_2}{L_2}\right)\mathrm{d}\zeta$$

$$(D_{11}, D_{12}, D_{22}) = \int_{-h/2}^{+h/2}\zeta^2\left(C'_{1111}\frac{L_2}{\alpha_2}\frac{\alpha_1}{L_1}, C'_{1122}, C'_{2222}\frac{L_1}{\alpha_1}\frac{\alpha_2}{L_2}\right)\mathrm{d}\zeta$$

$$(\tilde{B}_{11}, \tilde{B}_{12}, \tilde{B}_{22}) = \int_{-h/2}^{+h/2} f(\zeta)\left(C'_{1111}\frac{L_2}{\alpha_2}\frac{\alpha_1}{L_1}, C'_{1122}, C'_{2222}\frac{L_1}{\alpha_1}\frac{\alpha_2}{L_2}\right)\mathrm{d}\zeta$$

$$(\tilde{B}'_{11}, \tilde{B}'_{12}, \tilde{B}'_{22}) = \int_{-h/2}^{+h/2} f(\zeta)\left(C'_{1111}\frac{L_2}{\alpha_2}, C'_{1122}\frac{L_1}{\alpha_1}, C'_{2222}\frac{L_1^2}{\alpha_1^2}\frac{\alpha_2}{L_2}\right)\mathrm{d}\zeta$$

$$(d_{11}, d_{12}, d_{22}) = \int_{-h/2}^{+h/2}\zeta f(\zeta)\left(C'_{1111}\frac{L_2}{\alpha_2}\frac{\alpha_1}{L_1}, C'_{1122}, C'_{2222}\frac{L_1}{\alpha_1}\frac{\alpha_2}{L_2}\right)\mathrm{d}\zeta$$

$$(\tilde{D}_{11}, \tilde{D}_{12}, \tilde{D}_{22}) = \int_{-h/2}^{+h/2} f^2(\zeta)\left(C'_{1111}\frac{L_2}{\alpha_2}\frac{\alpha_1}{L_1}, C'_{1122}, C'_{2222}\frac{L_1}{\alpha_1}\frac{\alpha_2}{L_2}\right)\mathrm{d}\zeta$$

where we recall: $L_\beta = \alpha_\beta(1+\zeta/R_\beta)$. (33)

The global constitutive law, eqns (28) and (29), shows a coupling between membrane (displacement u), flexion (displacement w) and shear (rotation ω), for any material. This is due to the various curvatures of the shell and is well known.

CLASSICAL SHELL THEORY, PLUS FIRST-ORDER, THIRD-ORDER AND GENERALIZED SHEAR DEFORMATION THEORIES

Now, to use eqns (4)–(33) we only have to choose the "shear" function $f(\zeta)$. Several options are proposed and will be numerically evaluated in a future section. For an axisymmetric shell of general shape, these options are:

(a)
$$f(\zeta) = 0 \tag{34}$$

and eqns (4)–(33) give *the Kirchhoff–Love or classical thin shell theory*. Then the cross-sections remain plane and normal to the meridian.

(b)
$$f(\zeta) = \zeta \tag{35}$$

and eqns (4)–(33) give *the first-order shear deformation theory* (Reissner–Mindlin type theory). Then this theory allows the cross-sections to rotate relative to the meridian, the cross-sections remaining plane. This theory requires the use of a shear correction factor [correction of the shear modulus C_{3131} in eqn (32)] due to a rudimentary transverse shear stress distribution which, for example, is uniform for cylinders. The shear correction factor κ^2 has been evaluated in different ways.

(i) The right-hand side of eqn (32) is multiplied by a shear correction factor which is chosen to make the first-order shear deformation theory as accurate as possible for a specific problem. This technique was introduced for isotropic plates by Bollé (1947) in the manner of Timoshenko (1922) for beams, and by Mindlin (1951) using the asymptotic velocity of transverse waves in a plate (Rayleigh waves). Bollé found a correction factor equal to 5/6, while Mindlin found a correction factor which, obviously depends on the Poisson ratio.

(ii) In addition to the kinematic assumptions, one makes assumptions concerning the distribution of stresses and strains through the thickness of the shell. Then a variational principle has been employed to derive the constitutive law. This technique has been employed by Reissner (1945) and Naghdi (1963). With this technique assuming that the transverse shear stresses vary quadratically through the thickness for the first-order shear deformation theory, one found a shear correction factor equal to 5/6 for an isotropic plate.

(iii) For laminated plates, two shear correction factors have been determined by Whitney (1972). Explicit expressions are obtained for shear factors in cylindrical bending using the constitutive law of each layer in conjunction with the equilibrium equations which are integrated through the thickness of the laminate in order to obtain the transverse shear stresses. Integration constants are determined by satisfying the continuity of the transverse shear stresses at the various layer interfaces and the boundary conditions at the top and the bottom surfaces of the plate. For homogeneous plates, the method gives shear correction factors equal to 5/6.

The requirement of correction factors are a handicap to the use of the first-order shear deformation theory in multilayered structures. The shear correction κ^2 will be introduced in the right-hand side of eqn (32), i.e. the shear modulus C_{3131} is replaced by $\kappa^2 C_{3131}$.

(c)
$$f(\zeta) = \zeta(1 - 4\zeta^2/3h^2) \tag{36}$$

and eqns (4)–(33) give *the third-order shear deformation theory*. This function has been used by Bhimaraddi (1984) within another kind of kinematics for cylinders, and by Reddy and Liu (1985) for shallow-shells. This theory allows the cross-sections to rotate relative to the meridian and to warp into a third-order polynomial shape.

(d) $$f(\zeta) = (h/\pi)\sin(\pi\zeta/h) \quad (37)$$

and eqns (4)–(33) give *the generalized shear deformation theory*, where $\pi = 3.141592\ldots$ This last type of kinematics which is new, has been applied to plates by Touratier (1991). This theory allows the cross-sections to rotate relative to the meridian and to warp into a sine shape.

The correctness of the generalized shear deformation theory

In the absence of an exact three-dimensional form for the shell displacements, we are going to explain the choice of the kinematics in eqn (6) and the function given by the eqn (37), using three-dimensional plate considerations without membrane effects because these are well known and necessitate no justifications. Then, in eqn (6) membrane effects are represented by the term $(L_1/\alpha_1)u$. Cheng (1979) has presented a method for the solution of three-dimensional elasticity equations for the problem of thick plates. Through this method three governing differential equations, the well-known biharmonic equation $\nabla^2\nabla^2 w = -q/D$ (∇^2 is the two-dimensional Laplacian, D the bending rigidity, q the transverse load); a shear equation $(\nabla^2-(2p+1)^2\pi^2/h^2)s = 0$ (s is a shear function, p an integer); and a transcendental equation $(1/\nabla^2)(1-\sin(h\nabla)/h\nabla)H = 0$ (H is a stress function) are deduced from Navier's equations and are exact. We recall that the following discussion does not incorporate the membrane displacement $(L_1/\alpha_1)u$ in eqn (6). Then, we propose *building the shear-bending displacement field according to the above three-dimensional considerations and taking into account the motivations explained below eqn (6)*. In the displacement field corresponding to the biharmonic equation, we keep only the first term to avoid material behaviour dependence and higher-order derivatives: it is exactly the classical Kirchhoff Love displacement field which corresponds to $f(\zeta) = 0$, eqn (34), i.e. to $-(1/\alpha_1)\partial w/\partial\xi_1$ and w terms in eqn (6). Next, the shear equation gives an in-plane displacement field of which the thickness dependence is of the form $\sin((2p+1)\pi\zeta/h)$, *without any summation* on the nonzero odd integer $n = 2p+1$. Thus, we choose the particular solution $n = 1$ and introduce the multiplier factor h/π to build our generalized shear deformation theory so that $f(\zeta) = (h/\pi)\sin(\pi\zeta/h)$. This justifies the term $f(\zeta)\gamma(\xi_1, t)$ in eqn (6). Other particular choices are $n = 3$ or $n = 5$ or $n = 7,\ldots$ But we remark that $n = 1$ is the only solution to recover the first-order shear deformation theory by a truncation of the series development of the function $f(\zeta) = (h/\pi)\sin(\pi\zeta/h)$. In addition, the series development of $\sin(\pi\zeta/h)$ converges more rapidly than the series development of $\sin(n\pi\zeta/h)$ with $n = 3, 5,\ldots$ Note: all our computations in this paper are made *keeping the sine function intact* (without any series development truncation). Finally, the transcendental equation which gives a displacement field with a material dependence and higher-order derivatives, has not been taken into account.

All these remarks based on three-dimensional plate considerations have allowed us to build shear-bending displacements in eqn (6), or in eqn (4), and the generalized shear deformation theory represented by eqn (37), which is correct from the three-dimensional elasticity for thick plates.

Remark 1. To explain more traditionally the above eqn (4) and the proposed function $f(\zeta)$ given by eqn (37), we are going to expand in a power series of the thickness coordinate ζ the U_1 component of the displacement field in eqn (4). It is not necessary to examine the other components, since $U_2 = 0$ (axisymmetry) and $\varepsilon_{\zeta\zeta} = 0$ which implies $U_\zeta = w$. In principle, theories developed by this means can be made as accurately as desired simply by including a sufficient number of terms, see Lo *et al.* (1977) for example. Unfortunately, this method has serious limitations due to the great number of equations to be solved. Thus, the component U_1 of the displacement becomes, in the shear-bending case as in the preceding discussion (we recall that we want to explain only the shear-bending terms in the displacement field, the membrane term being classical):

$$U_1(\xi_1, \zeta, t) = \sum_{p=0}^{r} \zeta^{2p+1} \omega_1^{(2p+1)}(\xi_1, t). \tag{38}$$

So, between eqns (4) and (6) for the component U_1 without the membrane term, and the eqn (38), we have successively:

—from eqn (34), $f(\zeta) = 0$, then in eqn (38) taking into account eqns (4) and (6), we have:

$$\omega_1^{(1)} = -\frac{1}{\alpha_1}\frac{\partial w}{\partial \xi_1}, \quad \omega_1^{(3)} = \omega_1^{(5)} = \cdots = \omega_1^{(2p+1)} = \cdots = 0;$$

—from eqn (35), $f(\zeta) = \zeta$, then in eqn (38) taking into account eqns (4) and (6), we have:

$$\omega_1^{(1)} = \omega = \frac{-1}{\alpha_1}\frac{\partial w}{\partial \xi_1} + \gamma, \quad \omega_1^{(3)} = \omega_1^{(5)} = \cdots = \omega_1^{(2p+1)} = \cdots = 0;$$

—from eqn (36), $f(\zeta) = \zeta(1 - 4\zeta^2/3h^2)$ then in eqn (38) taking into account eqns (4) and (6), we have:

$$\omega_1^{(1)} = \omega = \frac{-1}{\alpha_1}\frac{\partial w}{\partial \xi_1} + \gamma, \quad \omega_1^{(3)} = -\frac{4}{3h^2}\gamma, \quad \omega_1^{(5)} = \cdots = \omega_1^{(2p+1)} = \cdots = 0;$$

from eqn (37),

$$f(\zeta) = \frac{h}{\pi}\sin\frac{\pi\zeta}{h} = \frac{h}{\pi}\sum_{p=0}^{r}(-1)^p\left(\frac{\pi\zeta}{h}\right)^{2p+1}\frac{1}{(2p+1)!},$$

then in eqn (38) taking into account eqns (4) and (6), we have:

$$\omega_1^{(1)} = \omega = -\frac{1}{\alpha_1}\frac{\partial w}{\partial \xi_1} + \gamma, \quad \omega_1^{(3)} = -\frac{\pi^2}{3!\,h^2}\gamma,$$

$$\omega_1^{(5)} = \frac{\pi^4}{5!\,h^4}\gamma, \ldots, \omega_1^{(2p+1)} = (-1)^p\frac{\pi^{2p}}{(2p+1)!\,h^{2p}}\gamma, \ldots$$

This shows the increasing refinement of the kinematics given by eqn (4) when we choose, successively, the function $f(\zeta)$ from eqn (34) to eqn (37).

Remark 2. It is interesting to note from the above considerations that:

—if $h \mapsto 0$, then $f(\zeta) = (h/\pi)\sin(\pi\zeta/h) \mapsto 0$, $\forall \zeta \in [-h/2, +h/2]$ and the Kirchhoff–Love theory is immediately recovered which is not the case with the first-order and the third-order shear deformation theories;

—the first-order shear deformation theory corresponds exactly to the first term of the series development of the generalized shear deformation theory which uses the sine function;

—the third-order shear deformation theory is of the same order as the third-order series development of the generalized shear deformation theory.

Remark 3. In eqn (8) if $f(\zeta)$ is defined by eqn (37), then $\mathrm{d}f/\mathrm{d}\zeta = \cos(\pi\zeta/h)$ and

(i) the transverse shear strain distribution is of high degree and a shear correction factor is not needed;

(ii) if $R_1 = \infty$ (cylinder cone), the boundary conditions for zero stress are exactly satisfied on the boundaries $\zeta = \pm h/2$;

(iii) for axisymmetric shells of general shape, these boundary conditions will be approximately satisfied if $h/[\pi R_1(1 \pm h/2R_1)] \ll 1$.

In conclusion, the three-dimensional thick plate's origin of the sine shear function (in absence of a three-dimensional exact solution for shells) and the infinity of terms in its polynomial representation allow us to hope that the corresponding theory will be accurate, without increasing the complexity compared to the first-order shear deformation theory.

SOME APPLICATIONS AND EXTENSIONS OF THE GENERAL AXISYMMETRIC SHELL THEORY

The preceding sections have been devoted to the axisymmetric shells of general shape. We now investigate some classic shapes of axisymmetric shells (Fig. 1).

(a) *Cylinder*

$$\xi_1 = s; \quad \xi_2 = \theta; \quad R_1 = \infty; \quad R_2 = R; \quad \alpha_1 = 1; \quad \alpha_2 = R. \tag{39}$$

(b) *Cone and circular plate*

$$\xi_1 = s; \quad \xi_2 = \theta; \quad R_1 = \infty; \quad R_2 = \frac{R_0}{\sin\varphi} + s\cot\varphi; \quad \alpha_1 = 1;$$

$$\alpha_2 = R = R_2 \sin\varphi = R_0 + s\cos\varphi. \tag{40}$$

For circular plates, we take $R_2 = \infty$, $\alpha_2 = R$.

(c) *Circular toroidal section*

$$\xi_1 = \varphi; \quad \xi_2 = \theta; \quad R_1(\text{constant}); \quad R_2 = \frac{R_c}{\sin\varphi} + R_1; \quad \alpha_1 = R_1;$$

$$\alpha_2 = R = R_2 \sin\varphi = R_c + R_1 \sin\varphi, \tag{41}$$

(d) *Spherical section*

$$\xi_1 = \varphi; \quad \xi_2 = \theta; \quad R_1(\text{constant}); \quad R_2 = R_1; \quad \alpha_1 = R_1; \quad \alpha_2 = R_2 \sin\varphi = R_1 \sin\varphi. \tag{42}$$

(e) *Paraboloid*

$$\xi_1 = \varphi; \quad \xi_2 = \theta; \quad R_1 = \frac{R_p}{\cos^3\varphi}; \quad R_2 = \frac{R_p}{\cos\varphi}; \quad \alpha_1 = R_1; \quad \alpha_2 = R = R_2 \sin\varphi. \tag{43}$$

(f) *Beam under traction-flexion in the plane* (ξ_1, ζ), $\vec{e}_1$ *is the beam axis*

$$\xi_1 = x_1; \quad \alpha_1 = \alpha_2 = 1; \quad R_1 = R_2 = \infty. \tag{44}$$

In this problem the elastic modulus must be taken equal to

$$C'_{1111} = E_1, \quad \text{Young's modulus following } \vec{e}_1.$$

The height of the beam is h and its width is unity. Finally, if

(i) $f(\zeta) = 0$, we obtain the Euler–Bernoulli beam theory;

(ii) $f(\zeta) = \zeta$, we obtain the Timoshenko beam theory, with a shear correction factor;

(iii) $f(\zeta) = \zeta(1-4\zeta^2/(3h^2))$, we obtain the Bickford (1982) theory; Levinson (1981) uses the same function $f(\zeta)$, but its theory cannot be deduced from the above model as the Levinson theory is variationally inconsistent.

(iv) $f(\zeta) = (h/\pi)\sin(\pi\zeta/h)$, gives the generalized shear deformation theory.

(g) *Curved beam in the plane* (ξ_1, ζ) ; ξ_1 *is following its axis*

$$\xi_1 = s;\quad R_1 = R_1(s);\quad R_2 = \infty;\quad \alpha_1 = 1;\quad \alpha_2 = 1. \tag{45}$$

(h) *Finite element approximations for a general shape* (Faye, to appear)

The kinematics in eqn (4) requires the C^1 continuity for the deflection w. In order to satisfy the field compatibility (to avoid the shear locking phenomena), i.e. the same degree of interpolation in the expression

$$\gamma = \omega + \frac{1}{\alpha_1}\frac{\partial w}{\partial \xi_1} \text{ for } \omega \text{ and } \frac{\partial w}{\partial \xi_1},$$

we suggest taking the classic Hermitian polynomial of degree three to interpolate w, and the classic Lagrange polynomial of degree two to interpolate both the membrane displacement u and the shear rotation ω. Then, the C^1 continuity is assured for w and u and ω are of C^0 continuity, as required. In linear elasticity, eqns (7), (15) and (27) will be used to build the stiffness matrix. An element such as that suggested above has three nodes with w^e, $\partial w^e/\partial \xi_1$, u^e, ω^e, as degrees of freedom at each end node of the element, and u^e, ω^e, as degrees of freedom at the central node of the element. Details for constructing such an element can be found in Zienkiewicz (1979), especially how to interpolate the curvature. The principal advantage of the present basic model, with $f(\zeta) = (h/\pi)\sin(\pi\zeta/h)$ for finite element approximation, is its lack of need of shear correction factors.

(i) *Generalization regarding an arbitrary shell*

We introduce in eqn (4) for the component U_2 of the displacement field an analog form to the component U_1 now with the generalized displacements u_β, ω_β, w ; $\beta = 1, 2$, and compute the strains using eqn (1). Then, all equations will be deduced from eqn (11), the local constitutive law being $\sigma_{ij} = C_{ijkl}e_{kl}$ (summation on k and $l = 1, 2, 3$) still with the assumption $\sigma_{\zeta\zeta} = 0$. Except for panels and cylinders, such a model is very complicated to use, and, today finite element approximations are more useful when simulating the behaviour of shells of arbitrary shapes, particularly for fully nonlinear analysis.

A NUMERICAL EVALUATION IN STATICS OF SEVERAL AXISYMMETRIC SHELL THEORIES

An evaluation of the generalized shear deformation theory is made in statics. The sample problem chosen is a simply-supported multilayered composite cylinder. Substitution of eqns (28)–(33) and (39) into eqn (24) yields the differential equations for the stable static loading problem of an orthotropic multilayered composite cylindrical shell under transverse normal pressure (internal pressure) p, expressed in terms of displacements of the midsurface of the shell ($g_{,s} = \mathrm{d}g/\mathrm{d}s$) :

$$A_{11}u_{,ss} + (\tilde{B}_{11} - B_{11})w_{,sss} + \frac{A'_{12}}{R}w_{,s} + \tilde{B}_{11}\omega_{,ss} = 0$$

$$(B'_{11} - \tilde{B}'_{11})u_{,sss} - \frac{A_{12}}{R}u_{,s} = (D_{11} - 2d_{11} + \tilde{D}_{11})w_{,ssss} + \left(\frac{2B_{12}}{R} - \frac{2\tilde{B}_{12}}{R} + \tilde{A}_{55}\right)w_{,ss}$$
$$- \frac{A'_{22}}{R^2}w + (d_{11} - \tilde{D}_{11})\omega_{,sss} + \left(\tilde{A}_{55} - \frac{\tilde{B}_{12}}{R}\right)\omega_{,s} + p = 0$$

$$\tilde{B}'_{11}u_{,ss} + (\tilde{D}_{11} - d_{11})w_{,sss} + \left(\frac{\tilde{B}_{12}}{R} - \tilde{A}_{55}\right)w_{,s} + \tilde{D}_{11}\omega_{,ss} - \tilde{A}_{55}\omega = 0 \tag{46}$$

and corresponding natural boundary conditions, in fact, eqn (25) with $R_1 = \infty$, $\tilde{m}_1 =$

$m_1 = 0, d\alpha_2/d\xi_1 = 0, \alpha_1 = 1$. Equations (46) are those of the generalized shear deformation theory for a cylindrical shell under an axisymmetric normal pressure. In particular, we have

—Kirchhoff–Love theory *with* $f(\zeta) = 0$ *and* $\tilde{B} = \tilde{B}' = \tilde{A} = \tilde{D} = d = 0$,
—First-order shear deformation theory *with* $f(\zeta) = \zeta$ *and* $B_{11} = \tilde{B}_{11} = \tilde{B}'_{11} = B'_{11}$, $d_{11} = D_{11} = \tilde{D}_{11}$, $B_{12} = \tilde{B}_{12}$.

We assume that the normal pressure p as sinusoidal. Then, the system of differential eqn (46) admits a solution of the form

$$u = U\cos\lambda s, \quad w = W\sin\lambda s, \quad \omega = \Omega\cos\lambda s \tag{47}$$

with

$$p = P\sin\lambda s \tag{48}$$

for a sinusoidal pressure. In eqns (47) and (48), l is the length of the cylinder, λ, U, W, Ω, P are constants which are to be determined, with the exception of P, by the boundary conditions. For a simply-supported cylindrical shell, these are [from eqns (9) and (25)]

$$w = 0, \quad M_{11} = \tilde{M}_{11} = 0, \quad N_{11} = 0. \tag{49}$$

Boundary conditions in eqn (49) are satisfied by eqn (47) with

$$\lambda = m\pi/l\,; \quad m = 1, 3, 5, \ldots, \text{an odd integer.} \tag{50}$$

Substituting eqns (47) and (48) into eqn (46) yields the following linear algebraic system which must be satisfied by the undetermined constants U, W and Ω:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{bmatrix} \begin{Bmatrix} U \\ W \\ \Omega \end{Bmatrix} = \begin{Bmatrix} 0 \\ -P \\ 0 \end{Bmatrix}, \tag{51}$$

where

$$a_{11} = -A_{11}\lambda^2; \quad a_{12} = (B_{11} - \tilde{B}_{11})\lambda^3 + \frac{A'_{12}}{R}\lambda; \quad a_{13} = -\tilde{B}_{11}\lambda^2$$

$$a_{22} = (2d_{11} - D_{11} - \tilde{D}_{11})\lambda^4 - \left(\frac{2B_{12}}{R} - \frac{2\tilde{B}_{12}}{R} + \tilde{A}_{55}\right)\lambda^2 - \frac{A'_{22}}{R^2}$$

$$a_{23} = (d_{11} - \tilde{D}_{11})\lambda^3 - \left(\tilde{A}_{55} - \frac{\tilde{B}_{12}}{R}\right)\lambda; \quad a_{33} = -\tilde{D}_{11}\lambda^2 - \tilde{A}_{55}. \tag{52}$$

This system is easy to solve.

Remarks:
—if the pressure is distributed following a single sine curve, then $m = 1$ in eqns (50), and only one solution is needed to obtain u, w, ω,
—if the pressure is uniformly distributed, then $m = 1, 3, 5, \ldots$ in eqn (50), and in eqn (51) $P = 4P_0/m\pi$. The solution of eqn (51) is obtained by the superposition of solutions corresponding to each $m = 1, 3, 5, \ldots$, until convergence of the solutions:

$$u = \sum_m U_m \cos(m\pi/l)s\,; \quad w = \sum_m W_m \sin(m\pi/l)s\,; \quad \omega = \sum_m \Omega_m \cos(m\pi/l)s, \tag{53}$$

—with Kirchhoff–Love theory, $a_{13} = a_{23} = a_{33} = 0$; $\tilde{B}_{11} = d_{11} = \tilde{D}_{11} = \tilde{B}_{12} = \tilde{A}_{55} = 0$

—with first-order shear deformation theory, $\tilde{B}_{11} = B_{11}$, $d_{11} = \tilde{D}_{11} = D_{11}$, $B_{12} = \tilde{B}_{12}$ and $\tilde{A}_{55}$ is computed with $f(\zeta) = \zeta$, and possibly a shear correction factor.

The preceding analysis is applied to the stable static loading of a simply-supported boron-epoxy composite circular cylindrical shell under internal radial pressure. The radial pressure is $P = 62.10$ MPa. The shell is assumed to be composed of four layers ($90°/0°/0°/90°$) of boron-epoxy composite of equal thickness. The fibers of the outer layers are then assumed to be oriented in the circumferential direction and the fibers in the inner layers are axially oriented. The following composite moduli and Poisson's ratio are used in the computation, Logan and Widera (1980):

$$E_1 = 241.5\ \text{GPa};\quad E_2 = 18.98\ \text{GPa};\quad E_3 = E_2$$

$$G_{23} = 3.45\ \text{GPa};\quad G_{31} = 5.18\ \text{GPa};\quad G_{12} = G_{31}$$

$$\nu_{23} = 0.25;\quad \nu_{31} = 0.24;\quad \nu_{12} = \nu_{31}.$$

The geometric properties of this thick cylindrical shell are:

$$R = 1.4097\ \text{m};\quad h = 0.2794\ \text{m},\quad l = 1\ \text{m or } 6.35\ \text{m}.$$

Then

$$R/h = 5.0454545;\quad l/h = 3.5790981 \text{ or } 22.727273;\quad R/l = 1.4097 \text{ or } 0.222.$$

Tables 1–5 show results obtained for displacements and stresses following the thickness of

Table 1. Maximum generalized displacements for a simply-supported circular cylindrical boron-epoxy layered ($90°/0°/0°/90°$) shell under internal sinusoidal pressure. Symbols SIN, CLT, FSDT, TSDT respectively designate the generalized shear deformation theory, the Kirchhoff–Love theory, the first-order shear deformation theory without correction factors and the third-order shear deformation theory; which correspond respectively to $f(\zeta) = (h/\pi)\sin(\pi\zeta/h)$; $f(\zeta) = 0$; $f(\zeta) = \zeta$; $f(1-4\zeta^2/3h^2)$

Theory	$w(l/2)$; in m	$u(0)$; in m	$\omega(0)$; in rd
SIN	2.7284×10^{-3}	2.8919×10^{-5}	-3.8432×10^{-3}
CLT	2.3211×10^{-3}	3.0421×10^{-5}	—
FSDT	2.6598×10^{-3}	2.9172×10^{-5}	-4.9253×10^{-3}
TSDT	2.7247×10^{-3}	2.8932×10^{-5}	-3.9622×10^{-3}

Table 2. Distribution of the $\sigma_{1\zeta}$ transverse shear stress throughout the thickness of a simply-supported circular cylindrical boron-epoxy layered ($90°/0°/0°/90°$) shell under internal sinusoidal pressure: SIN $\sigma_{1\zeta}$ is the generalized shear deformation theory; CLT $\sigma_{1\zeta}$ is the Kirchhoff–Love theory; FSDT $\sigma_{1\zeta}$ is the first-order shear deformation theory; TSDT $\sigma_{1\zeta}$ is the third-order shear deformation theory. The distribution of the transverse shear stress is computed at the end $\xi_1 = s = 0$ of the shell. Symbols SIN, CLT, FSDT, TSDT correspond to $f(\zeta) = (h/\pi)\sin(\pi\zeta/h)$; $f(\zeta) = 0$; $f(\zeta) = \zeta$; $f(\zeta) = \zeta(1-4\zeta^2/3h^2)$

	SIN $\sigma_{1\zeta}(0,\zeta)$ in MPa	CLT $\sigma_{1\zeta}(0,\zeta)$ in MPa	FSDT $\sigma_{1\zeta}(0,\zeta)$ in MPa	TSDT $\sigma_{1\zeta}(0,\zeta)$ in MPa
$\zeta = 0$ (meridian)	24.492	0	17.770	23.816
$\zeta = h/4$ (interface)	17.318	0	17.770	17.862
	11.534	0	11.835	11.896
$\zeta = h/2$ (external face)	0	0	11.835	0

Table 3. Distribution of the U_1 meridional displacement throughout the thickness of a simply-supported cylindrical boron-epoxy ($90°/0°/0°/90°$) layered shell under internal sinusoidal pressure. Symbols SIN, CLT, FSDT, TSDT respectively represent the generalized shear deformation theory, the Kirchhoff–Love theory, the first-order shear deformation theory and the third-order shear deformation theory, which correspond to $f(\zeta) = (h/\pi)\sin(\pi\zeta/h)$; $f(\zeta) = 0$; $f(\zeta) = \zeta$; $f(\zeta) = \zeta(1-4\zeta^2/3h^2)$

	SIN $U_1(0,\zeta)$ in m	CLT $U_1(0,\zeta)$ in m	FSDT $U_1(0,\zeta)$ in m	TSDT $U_1(0,\zeta)$ in m
$\zeta = 0$ (meridian)	2.8919×10^{-5}	3.0421×10^{-5}	2.9172×10^{-5}	2.8932×10^{-5}
$\zeta = h/4$ (interface)	-2.7245×10^{-4}	-4.7892×10^{-4}	-3.1486×10^{-4}	-2.7459×10^{-4}
$\zeta = h/2$ (external face)	-7.4800×10^{-4}	-9.8826×10^{-4}	-6.5890×10^{-4}	-7.3868×10^{-4}

Table 4. Distribution of the meridional stress σ_{11} for a simply-supported boron-epoxy (90°/0°/0°/90°) layered shell under internal sinusoidal pressure. Symbols SIN, CLT, FSDT, TSDT respectively represent the generalized shear deformation theory, the Kirchhoff–Love theory, the first-order shear deformation theory and the third-order shear deformation theory, which correspond to $f(\zeta) = (h/\pi)\sin(\pi\zeta/h)$; $f(\zeta) = 0$; $f(\zeta) = \zeta$; $f(\zeta) = \zeta(1-4\zeta^2/3h^2)$

	SIN $\sigma_{11}(l/2,\zeta)$ in MPa	CLT $\sigma_{11}(l/2,\zeta)$ in MPa	FSDT $\sigma_{11}(l/2,\zeta)$ in MPa	TSDT $\sigma_{11}(l/2,\zeta)$ in MPa
$\zeta = 0$ (meridian)	−13.184	−15.651	−13.599	−13.206
$\zeta = h/4$ (interface)	216.08	372.18	248.20	217.70
	24.757	35.865	27.086	24.875
$\zeta = h/2$ (external face)	52.862	66.050	47.322	52.29

Table 5. Distribution of the σ_{22} circumferential stress throughout the thickness of a simply-supported boron-epoxy (90°/0°/0°/90°) layered shell under internal sinusoidal pressure. Symbols SIN, CLT, FSDT, TSDT respectively represent the generalized shear deformation theory, the Kirchhoff–Love theory, the first-order shear deformation theory and the third-order shear deformation theory, which correspond to $f(\zeta) = (h/\pi)\sin(\pi\zeta/h)$; $f(\zeta) = 0$; $f(\zeta) = \zeta$; $f(\zeta) = \zeta(1-4\zeta^2/3h^2)$

	SIN $\sigma_{22}(l/2,\zeta)$ in MPa	CLT $\sigma_{22}(l/2,\zeta)$ in MPa	FSDT $\sigma_{22}(l/2,\zeta)$ in MPa	TSDT $\sigma_{22}(l/2,\zeta)$ in MPa
$\zeta = 0$ (meridian)	36.485	30.955	35.554	36.43
$\zeta = h/4$ (interface)	39.076	36.796	38.802	39.059
	451.28	387.47	440.64	450.71
$\zeta = h/2$ (external face)	437.95	377.63	425.93	437.24

a multilayered cylindrical short shell with $l = 1$ m. In these tables, symbols SIN, CLT, FSDT, TSDT, respectively identify:

SIN: the present generalized shear deformation theory with $f(\zeta) = (h/\pi)\sin(\pi\zeta/h)$,

CLT: the classical laminated theory, i.e. Kirchhoff–Love theory for laminated, with $f(\zeta) = 0$,

FSDT: the first-order shear deformation theory *without shear correction factor* and $f(\zeta) = \zeta$,

TSDT: the third-order shear deformation theory such as:

$$f(\zeta) = \zeta(1-4\zeta^2/3h^2).$$

In Tables 1–5, displacement distributions are deduced from eqns (4), (39), (47) and (51) and stress distributions from eqns (27), (7), (39), (47) and (51).

Discussion of the results

Comparisons of SIN, CLT, FSDT and TSDT theories in the absence of an exact three-dimensional elasticity solution are made. Results in Tables 1–5 show that:

—The CLT solution is not applicable to the problem in question. In fact, errors are large, except for the membrane displacement which gives a deviation of 5% between CLT and SIN solutions. All stresses become inaccurate when using the CLT theory.

—The FSDT solution gives a good approximation of transverse displacement, membrane displacement, circumferential stress; but a large error is seen, compared with the SIN solution for shear rotation ω, meridional displacement U_1, transverse shear stress $\sigma_{1\zeta}$ and meridional stress σ_{11}.

—The TSDT solution involves a maximum deviation of 3.2% in comparison with the SIN solution for shear rotation and transverse shear stress.

All inaccuracies observed on the short cylinder using CLT and FSDT are due to transverse shear effects. To confirm this observation, it is sufficient to consider a simply-supported moderately long cylinder: we have chosen $l = 6.35$ m, as in Logan and Widera (1980). Then the maximum deviation between the SIN solution and CLT or FSDT solutions for generalized displacements u and w is below 1%. For the meridional stress $\sigma_{11}(l/2,\zeta)$, the maximum deviation between the SIN and CLT theories is around 9%; and around 2%

between the SIN and FSDT theories, at the interface $\zeta = h/4$ (SIN value = 3.48 MPa, the maximum meridional stress is 10.5 MPa for all theories). The maximum transverse shear stress $\sigma_{1\zeta}$ becomes very small in this example and is not significant: $\sigma_{1\zeta}(0,0) = 0.21$ MPa with the SIN solution and $\sigma_{1\zeta}(0,0) = 0.14$ MPa with the FSDT solution. This is why all theories give the same results for maximum generalized displacements for a moderately long layered cylindrical shell: $w(l/2) \sim 3.361 \times 10^{-3}$ m; $u(0) \sim 1.713 \times 10^{-4}$ m.

Finally, as the generalized shear deformation theory proposed is simple, it is suggested it is used for any shell problems (thin and moderately thick shells with or without composite materials). Anyway, a shear deformation theory has to be chosen for shell problems in which the shear is significant, i.e. for example with moderately thick shells, laminated shells (thin or thick) because they exhibit much lower strength in the transverse directions and at the ply interfaces, thus being particularly susceptible to matrix cracking and delaminations. Finally, shear deformation theories have to be chosen for any shell type in wave propagation phenomenon analysis.

CONCLUDING REMARKS

In this paper, a generalized shear deformation theory has been proposed for moderately thick multilayered axisymmetric shells without any assumption other than neglecting the transverse normal strain. The reduction of the three-dimensional problem to the bidimensional one is accomplished assuming a displacement field which allows sine variations throughout the thickness of the shell for the U_1 meridional displacement, and a constant value for the U_ζ radial displacement. The shear in the proposed theory is represented by trigonometric functions and does not require the introduction of shear correction factors. The boundary value problem is solved by the principle of virtual power, and is fully general when introducing a function $f(\zeta)$ in the kinematics [eqn (4)] which allows us to systematically obtain all required results (kinematics, strains, equilibrium equations, natural boundary conditions, stresses, constitutive law) for the Kirchhoff–Love theory where $f(\zeta) = 0$, the first-order shear deformation theory where $f(\zeta) = \zeta$, the third-order shear deformation theory where $f(\zeta) = \zeta(1-4\zeta^2/(3h^2))$ and finally, a generalized shear deformation theory where $f(\zeta) = (h/\pi)\sin(\pi\zeta/h)$, as proposed in this paper and which is new. The theory is presented for an arbitrary shape of axisymmetric multilayered shells. Further classic shapes are mentioned as examples, and the derivation of a finite element approximation for an arbitrary shape is indicated, as well as the extension to an arbitrary shell without axisymmetry. A numerical test of comparison for a multilayered cylindrical shell is given between the Kirchhoff–Love theory, the first-order shear deformation theory, the third-order shear deformation theory and the generalized shear deformation theory. From a convergence point of view and in the absence of an exact three-dimensional elasticity solution the numerical trend indicates as in plates that the better reference solution between all theories, seems to be that of the generalized shear deformation theory. In fact, this behaviour has been observed between the same theories as in this paper, but for plates by comparison with the exact three-dimensional solution which exists, Touratier (1991). Future work will be turned toward the edge effects in the objective of the sizing of structures.

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